A Kazhdan group with an infinite outer automorphism group

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Abstract. D. Kazhdan has introduced in 1967 the Property (T) for local compact groups (see [3]). In this article we prove that for $n \geq 3$ and $m \in \mathbb{N}$ the group $SL_n(\mathbf{K}) \ltimes \mathcal{M}_{n,m}(\mathbf{K})$ is a Kazhdan group having the outer automorphism group infinite.

Definition 1. *([1]) Let $(\pi, \mathcal{H})$ be a unitary representation of a topological group G.*

(i) For a subset Q of G and real number $\varepsilon > 0$, a vector $\xi \in \mathcal{H}$ is (Q, ε)-invariant if :

$$sup_{x \in Q} ||\pi(x)\xi - \xi|| < \varepsilon ||\xi||.$$

(ii) The representation $(\pi, \mathcal{H})$ almost has invariant vectors if it has (Q, ε) - invariant vectors for every compact subset Q of G and every $\varepsilon > 0$. If this holds, we write $1_G \prec \pi$.

(iii) The representation $(\pi, \mathcal{H})$ has non - zero invariant vectors if there exists $\xi \neq 0$ in $\mathcal{H}$ such that $\pi(x)\xi = \xi$ for all $g \in G$. If this holds, we write $1_G \subset \pi$.

Definition 2. *([3]) Let G be a topological group.*

G has Kazhdan's Property (T), or is a Kazhdan group, if there exists a compact subset Q of G and $\varepsilon > 0$ such that, whenever a unitary representation π of G has a (Q, ε) - invariant vector, then π has a non-zero invariant vector.

Proposition 3. *([1]) Let G be a topological group.The following statements are equivalent:*

(i) G has Kazhdan's Property(T);

(ii) whenever a unitary representation $(\pi, \mathcal{H})$ of G weakly contains 1_G, it contains 1_G (in symbols: $1_G \prec \pi$ implies $1_G \subset \pi$).

Definition 4. *Let $\boldsymbol{K}$ be a field. An absolute value on $\boldsymbol{K}$ is a real - valued function $x \to |x|$ such that, for all x and y in $\boldsymbol{K}$:*

(i) $|x| \geq 0$ and $|x| = 0 \Leftrightarrow x = 0$

(ii)$|xy| = |x||y|$

(iii)$|x + y| \leq |x| + |y|$.

An absolute value defines a topology on $\boldsymbol{K}$ given by the metric

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$$d(x,\ y) = |x - y|.$$

Definition 5. *A field* $\boldsymbol{K}$ *is a local field if* $\boldsymbol{K}$ *can be equipped with an absolute value for which* $\boldsymbol{K}$ *is locally compact and not discrete.*

Example 6. $\boldsymbol{K} = \mathbb{R}$ *and* $\boldsymbol{K} = \mathbb{C}$ *with the usual absolute value are local fields.*

Example 7. *([1] and [2]) Groups with Property (T):*
a) Compact groups, $SL_n(\mathbb{Z})$ *for* $n \geq 3$.
b) $SL_n(\boldsymbol{K})$ *for* $n \geq 3$ *and* $\boldsymbol{K}$ *a local field.*

Lemma 8. *(Mautner's lemma)([1])*
Let G *be a topological group, and let* $(\pi, \mathcal{H})$ *be a unitary representation of* G. *Let* $x \in G$ *and assume that there exists a net* $(y_i)_i$ *in* G *such that* $\lim_i y_i x y_i^{-1} = e$. *If* ξ *is a vector in* $\mathcal{H}$ *which is fixed by* y_i *for all* i, *then* ξ *is fixed by* x.

Theorem 9. *Let* $\boldsymbol{K}$ *be a local field. The group* $SL_n(\boldsymbol{K})$ *acts on* $\mathcal{M}_{n,m}(\boldsymbol{K})$ *by left multiplication* $(g, A) \to gA$, $g \in SL_n(\boldsymbol{K})$ *and* $A \in \mathcal{M}_{n,m}(\boldsymbol{K})$.
Then the semi - direct product $SL_n(\boldsymbol{K}) \ltimes \mathcal{M}_{n,m}(\boldsymbol{K})$ *has Property (T) for* $(\forall) n \geq 3$ *and* $(\forall) m \in \mathbb{N}$.

Proof. Let $(\pi, \mathcal{H})$ be a unitary representation of G $= SL_n(\mathbf{K}) \ltimes \mathcal{M}_{n,m}(\mathbf{K})$ almost having invariant vectors. Since $SL_n(\mathbf{K})$ has Property (T), there exists a non - zero vector $\xi \in \mathcal{H}$ which is $SL_n(\mathbf{K})$ - invariant.

Since $\mathbf{K}$ is non - discret, there exists a net $(\lambda_i)_i$ in $\mathbf{K}$ with $\lambda_i \neq 0$ and such that $\lim_i \lambda_i = 0$.

Let $\Delta_{pq}(x) \in \mathcal{M}_{n,m}(\mathbf{K})$ the matrix with x as (p,q) - entry and 0 elsewhere and $(A_i)_{\alpha\beta} \in SL_n(\mathbf{K})$ the matrix:

$$(A_i)_{\alpha,\beta} = \begin{cases} \lambda_i & if\ \alpha = \beta\ and\ \alpha = p \\ \lambda_i^{-1} & if\ \alpha = \beta\ and\ \alpha = (p+1) mod(n+1) + [p/n] \\ 1 & if\ \alpha = \beta\ and\ \alpha \notin \{p, (p+1) mod(n+1) + [p/n]\} \\ 0 & if\ \alpha \neq \beta \end{cases} \tag{1}$$

$\Rightarrow A_i \Delta_{pq}(x) = \delta_{pq}(\lambda_i x)$, where $\delta_{pq}(\lambda_i x) \in \mathcal{M}_{n,m}(\mathbf{K})$ is the matrix with $\lambda_i x$ as (p, q) - entry and 0 elsewhere.

Then $\lim_i A_i \Delta_{pq}(x) = 0_{n,m}$.

Since in G we have

$$(A_i, 0_{n,m})(I_n, \Delta_{pq}(x))(A_i, 0_{n,m})^{-1} = (I_n, A_i \Delta_{pq}(x))$$

and since $\xi \in \mathcal{H}$ is $(A_i, 0_{n,m})$ - invariant $\Rightarrow$
$\Rightarrow$ from Mautner's Lemma that ξ is $\Delta_{pq}(x)$ - invariant.
Since $\Delta_{pq}(x)$ generates the group $\mathcal{M}_{n,m}(\mathbf{K}) \Rightarrow \xi$ is G - invariant and G has Property (T). $\square$

Corollary 10. *The groups $SL_n(\boldsymbol{K}) \ltimes \boldsymbol{K}^n$ and $SL_n(\mathbb{R}) \ltimes \mathcal{M}_n(\mathbb{R})$ has Property (T),* $(\forall)n \geq 3$.

Proposition 11. *For $\delta \in SL_n(\mathbb{Z})$, let $S_\delta : \Gamma \to \Gamma$, $S_\delta((\alpha, A)) = (\alpha, A\delta), (\forall)(\alpha, A) \in \Gamma$. Then:*
a) $S_\delta \in Aut(\Gamma)$.
b) $\Phi : SL_n(\mathbb{Z}) \to Aut(\Gamma)$, $\Phi(\delta) = S_\delta$ is a group homomorphism.
c)$S_\delta \in Int(\Gamma)$ if and only if $\delta \in \{\pm I\}$. In particular, the outer automorphism of Γ is infinit.

Proof. a) $S_\delta((\alpha_1, A_1) \cdot (\alpha_2, A_2)) = S_\delta((\alpha_1, A_1)) \cdot S_\delta((\alpha_2, A_2)) \Leftrightarrow$
$\Leftrightarrow S_\delta((\alpha_1\alpha_2, A_1 + \alpha_1 A_2)) = (\alpha_1, A_1\delta) \cdot (\alpha_2, A_2\delta) \Leftrightarrow$
$\Leftrightarrow (\alpha_1\alpha_2, (A_1 + \alpha_1 A_2)\delta) = (\alpha_1\alpha_2, A_1\delta + \alpha_1 A_2\delta)$
Analogous $S_{\delta^{-1}}$ is morfism and $S_\delta \cdot S_{\delta^{-1}} = S_{\delta^{-1}} \cdot S_\delta = I_\Gamma$.
b)$\Phi(\delta_1 \cdot \delta_2) = \Phi(\delta_1) \cdot \Phi(\delta_2) \Leftrightarrow S_{\delta_1 \cdot \delta_2} = S_{\delta_1} \cdot S_{\delta_2}$.
c) Assume that $S_\delta \in Int(\Gamma) \Rightarrow (\exists)(\alpha_0, A_0) \in \Gamma$ such that
$S_\delta((\alpha, A)) = (\alpha_0, A_0)(\alpha, A)(\alpha_0, A_0)^{-1}, (\forall)(\alpha, A) \in \Gamma$.
$\Rightarrow (\alpha, A\delta) = (\alpha_0\alpha\alpha_0^{-1}, A_0 + \alpha_0 A - \alpha_0\alpha\alpha_0^{-1} A_0) \Rightarrow$
$\Rightarrow$ i) $\alpha = \alpha_0\alpha\alpha_0^{-1}, (\forall)\alpha \in SL_n(\mathbb{Z}) \Rightarrow \alpha \in \{\pm I_n\}$
$\Rightarrow$ ii) $A\delta = A_0 \pm A - \alpha A_0, (\forall)\alpha \in SL_n(\mathbb{Z}), (\forall)A \in \mathcal{M}_n(\mathbb{Z}) \Rightarrow A_0 = 0_n$ and $\delta = \pm I_n$.

$$\Rightarrow Out(\Gamma) = {Aut(\Gamma)} \Big/ {Int(\Gamma)} \quad is \ \ infinite.$$

$\square$

References

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